An overview of basic mathematics

Christian Uldal Graulund Patrick Bahr

May 20, 2022

1 Naive set theory

A set is a collection of objects. An object a belonging to A, is called an *element* of A. We write $a \in A$ to denote that a belongs to A.

We are especially interested in certain sets of numbers:

- The natural numbers:

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

- The integers (whole numbers):

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

- The rational numbers:

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z}, b \in \mathbb{N} \right\}$$

  Two rational numbers $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$ are equal, if $a_1 \cdot b_2 = b_1 \cdot a_2$.

- The real numbers denoted $\mathbb{R}$. It is difficult to give a short and easy to understand description of real numbers. Naively we could say that real numbers are all the numbers missing from the rational numbers. A more correct definition would be "all the numbers that can be approximated arbitrarily precise by a sequence of rational numbers", or "the algebraic completion of the rational numbers". Neither of these are easy to get to, so we shall stick with the first, naive, description.[1]

2 Rules for addition and multiplication

Addition and multiplication satisfy a number of rules that allow us to simplify arithmetic expressions.

[1] In reality, these definition are not *that* hard to get to. An interested student would be able to get to them in a couple of weeks

1. Associativity: Both addition and multiplication are *associative*, i.e. for all $a, b, c \in \mathbb{R}$, we have that

$$a + (b + c) = (a + b) + c$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

   For example, $2 + (3 + 4) = (2 + 3) + 4 = 9$

2. Commutativity: Both addition and multiplication are *commutative*, i.e. for all $a, b \in \mathbb{R}$, we have that

$$a + b = b + a$$
$$a \cdot b = b \cdot a$$

   For example, $4 + 2 = 2 + 4 = 6$.

3. Distributivity: Multiplication *distributes* over addition, i.e. for all $a, b, c \in \mathbb{R}$, we have that

$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$(b + c) \cdot a = b \cdot a + c \cdot a$$

4. Additive identity 0, i.e. for all $a \in \mathbb{R}$

$$a + 0 = a \qquad \text{and} \qquad 0 + a = a$$

5. Multiplicative identity 1, i.e. for all $a \in \mathbb{R}$

$$a \cdot 1 = a \qquad \text{and} \qquad 1 \cdot a = a$$

6. Additive inverse: Every $a \in \mathbb{R}$ has an *additive inverse* $-a$ such that

$$a + (-a) = 0$$

   We also write $a - b$ instead of $a + (-b)$.

7. Multiplicative inverse: Every non-zero $a \in \mathbb{R}$ has a *multiplicative inverse* a^{-1} (also written $\frac{1}{a}$) such that

$$a \cdot a^{-1} = 1$$

   We also write $\frac{a}{b}$ or a/b ("a divided by b") instead of $a \cdot b^{-1}$.

3 Fractions

A *fraction* is a powerful way to describe division of two numbers. Let $a, b \in \mathbb{R}$ with $b \neq 0$. We then write

$$\frac{a}{b}$$

We call a the *numerator* and b the *denominator*.

3.1 Rules for fractions

1. Addition and subtraction with common denominator:

$$\frac{a}{b} \pm \frac{c}{b} = \frac{a \pm c}{b}$$

   where $a, b, c \in \mathbb{R}$ with $b \neq 0$.

2. The general formula for addition and subtraction is:

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$$

   where $a, b, c, d \in \mathbb{R}$ with $b, d \neq 0$.

3. General formula for multiplication:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

   where $a, b, c, d \in \mathbb{R}$ with $b, d \neq 0$.

4. General formula for division:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

   where $a, b, c, d \in \mathbb{R}$ with $b, c, d \neq 0$.

3.2 Simplifying fractions

A fraction is in *simplest form* when the numerator and denominator have no common factors (or divisors) other than 1.

Example: The rational number $\frac{2}{4}$ is not in its simplest form. But $\frac{1}{2}$ is.

Every rational number is equal to one in the simplest from. The following rules can be used to simplify fractions

1. If the numerator and the denominator are equal, $\frac{a}{a} = 1$

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = ...$$

2. A fraction $\frac{a}{1}$ with denominator 1 is written simply as a.

$$\frac{5}{1} = 5 \qquad \frac{24}{1} = 24 \qquad \frac{-7}{1} = -7$$

3. A fraction $\frac{0}{a}$ is equal to 0.

$$\frac{0}{8} = 0 \qquad \frac{0}{56} = 0 \qquad \frac{0}{-11} = 0$$

4. **Note:** A fraction $\frac{a}{0}$ is undefined

$$\frac{7}{0} \text{ is undefined} \qquad \frac{-18}{0} \text{ is undefined}$$

5. Let a, m, n be real numbers with $a, n \neq 0$. Then

$$\frac{a \cdot m}{a \cdot n} = \frac{m}{n}$$

4 Exponentiation

Let $n \in \mathbb{N}$ be a natural numbers, and $a \in \mathbb{R}$ a real. Define

$$a^n = \overbrace{a \cdot a \cdot \ldots \cdot a}^{\text{n times}}, \quad a^0 = 1, \quad a^{-n} = \frac{1}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{\text{n times}}}$$

We call a the *base number* and n the *exponent*.

NOTE: This is the *definition* of what exponents mean. Hence any time we see a^b we can substitute the above if we want.

4.1 Rules for exponentiation:

In the following, let $a, b \in \mathbb{R}$, while $m, n \in \mathbb{N}$.

1. Multiplication with common base number:

$$a^n \cdot a^m = a^{n+m}$$

2. Division with common base number:

$$\frac{a^n}{a^m} = a^{n-m}$$

3. Multiplication with different base, same exponent:

$$a^n \cdot b^n = (a \cdot b)^n$$

4. Division with different base, same exponent:

$$\frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$$

5. Exponentiation of exponentiation :

$$(a^n)^m = a^{n \cdot m}$$

4.2 Rational exponents

In the following, let $a \in \mathbb{R}_+$, and $m, n \in \mathbb{N}$.

Definition: We define exponentiation with rational exponents as:

$$a^{m/n} = \sqrt[n]{a^m}$$

The above rules for natural exponents are also valid for rational exponents

4.3 Logarithms

The *logarithm* is the inverse of exponentiation. More precisely, if $b^x = y$, then the logarithm of y to base b is x, which we write as $\log_b y = x$. In other words, the logarithm gives us the exponent x to which we must raise the base b so that we obtain the result y.

4.4 Rules for logarithms

In the following, let $a \in \mathbb{R}$, and $b, c, x, y \in \mathbb{R}_+$ such that $b \neq 1$ and $c \neq 1$.

1. Addition with common base:

$$\log_b(x \cdot y) = \log_b x + \log_b y$$

2. Subtraction with common base:

$$\log_b \left(\frac{x}{y} \right) = \log_b x - \log_b y$$

3. Multiplication:

$$\log_b \left(x^a \right) = a \log_b x$$

4. Division:

$$\log_c x = \frac{\log_b x}{\log_b c}$$

5. Exponentiation:

$$b^{\log_b y} = y$$

5 Binomial theorems for squares

Let $a, b \in \mathbb{R}$ be given. Then we have the following:

First formula:

$$(a+b)^2 = a^2 + 2ab + b^2$$

Second formula:

$$(a-b)^2 = a^2 - 2ab + b^2$$

Third formula:

$$(a+b)(a-b) = a^2 - b^2$$

6 Equations

An equation is a *predicate* with one or more free variables (typically denoted by $x, y, z, \ldots$)

The goal in solving an equation is to get concrete values for the free variables, such that the equations holds for these concrete values.

We talk about the *variable set* as the set each variable comes from. For instance, which values can x take in the following?

$$\frac{1}{x} = \sqrt{x} \quad \text{and} \quad \frac{x+5}{x+2} = \frac{6}{2x+4}$$

The basic technique is applying **identical** algebraic transformations on both sides of the equality sign.

We denote the set of *all solutions* as the solution set. There may be one, several, none or infinitely many solutions to any given equation.

6.1 An example

We look at the equation:

$$\frac{x}{\sqrt{4x-1}} = \frac{1}{2}$$

What values can x take?

$$4x - 1 > 0 \Leftrightarrow x > \frac{1}{4}$$

We can now do the following transformations:

$$\begin{aligned}
\frac{x}{\sqrt{4x-1}} = \frac{1}{2} \Rightarrow \frac{x^2}{4x-1} &= \frac{1}{4} \\
&\Leftrightarrow x^2 = \frac{1}{4}(4x-1) \\
&\Leftrightarrow 4x^2 = 4x - 1 \\
&\Leftrightarrow 4x^2 - 4x + 1 = 0 \\
&\Leftrightarrow (2x-1)^2 = 0 \\
&\Leftrightarrow 2x - 1 = 0 \\
&\Leftrightarrow 2x = 1 \\
&\Leftrightarrow x = \frac{1}{2}
\end{aligned}$$

We note that $x = \frac{1}{2} > \frac{1}{4}$, and hence we can accept the solution.

6.2 Absolute value

The *absolute value* of a real number a, written as $|a|$, is defined as follows: If a is negative, then $|a| = -a$, otherwise $|a| = a$. That is, $|a|$ is always positive.

Examples:

1. $|3| = 3$ and $|-3| = 3$.
2. Consider the equation $|x + 5| = 2$. What values can x take?

   By the definition of absolute value, we know that

   $$x + 5 = 2 \text{ or } x + 5 = -2$$

   Therefore,

   $$x = -3 \text{ or } x = -7$$

7 Systems of linear equations:

Equations can be viewed as *restrictions* on the free variables $x, y, z, \ldots$. Given two equations:

$$y = -2x + 3 \quad \text{and} \quad y = -\frac{3}{2}x + 2$$

We want to find values for x and y such that both equations are solved **simultaneously**.

The basic technique for systems of equations, is the do *substitution* from one equation to the other. We isolate one variable, and the substitute it into the other. In the above example the variable y is isolated in both equations and hence we can substitute in from one to the other:

$$\begin{aligned} y = -2x + 3 &\Leftrightarrow -\frac{3}{2}x + 2 = -2x + 3 \\ &\Leftrightarrow -\frac{3}{2}x = -2x + 1 \\ &\Leftrightarrow -3x = -4x + 2 \\ &\Leftrightarrow x = 2 \end{aligned}$$

We can now substitute the (concrete) value for x into the second equation, to get a (concrete) value for y:

$$\begin{aligned} y = -\frac{3}{2}x + 2 &\Leftrightarrow y = -\frac{3}{2} \cdot 2 + 2 \\ &\Leftrightarrow y = -3 + 2 \\ &\Leftrightarrow y = -1 \end{aligned}$$

To check our solution we insert the concrete values into our two equations, and see if the reduce to identities:

$$y = -2x + 3 \Leftrightarrow -1 = -2 \cdot 2 + 3 \Leftrightarrow -1 = -4 + 3$$

and

$$y = -\frac{3}{2}x + 2 \Leftrightarrow -1 = -\frac{3}{2} \cdot 2 + 2 \Leftrightarrow -1 = -3 + 2$$

Since the holds we accept the solution $x = 2, y = -1$.

8 Quadratic equations

A quadratic equation is a equation on the form:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

Theorem: The equation:

$$ax^2 + bx + c = 0, \quad a \neq 0$$

has the solutions:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 - 4ac \geq 0$.

8.1 Substitution in quadratic equations

Sometimes we can rearrange higher order equations to look like quadratic equations. We call this principle **substitution**.

In general, we substitute one variable for another the reduce *the degree*[2] of the equation. If we look at

$$x^4 - x^2 - 12 = 0, \quad \text{and} \quad 2x^4 - 5x^2 + \frac{9}{8} = 0$$

we see that if we choose $z = x^2$ we can change these to:

$$z^2 - z - 12 = 0, \quad \text{and} \quad 2z^2 - 5z + \frac{9}{8} = 0$$

Both equations are now quadratic, and hence we can use the theorem for quadratic equations. We can then obtain a solution for z and use the solution to get a solution for x.

Note that we will often get more then two solutions when doing equations of higher degrees. In general, for a n degree equation we can get up to n solution.

[2] The degree of an equations of the highest exponentiation of a free variable. In the examples the degree is 4